PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Phil. Trans. R. Soc. Lond. A 1952 **244**, 357-384
doi: 10.1098/rsta.1952.0009

ON TURBULENCE CAUSED BY THERMAL INSTABILITY

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(*Received* 18 *September* 1951)

In this paper a statistical theory of turbulence in an incompressible fluid caused by the joint effects of gravity, and thermal instability, is developed. The mathematical theory is based on the equations of continuity and heat conduction and the Boussinesq form of the equations of motion in which the variations of density (resulting from the variations in temperature) are taken into account only in so far as they modify the action of gravity. By restricting oneself to a portion of the fluid far from the bounding surfaces one can treat the turbulence as approximately homogeneous and axisymmetric and use the theory of axisymmetric vectors and tensors recently developed by the writer (Chandrasekhar 1950*a*). A number of correlations between the various field quantities (such as the velocity components, fluctuations in temperature, etc.) at two different points in the medium are defined; and a closed system of equations for the defining scalars are derived for the case when the non-linear terms in the equations of motion and heat conduction can be neglected and a constant mean adverse temperature gradient is maintained. Under stationary conditions when the time derivatives of the various correlations are zero, there is an exact balance between the dissipation of kinetic energy by viscosity and the liberation of potential energy by gravity.

A fundamental set of solutions of the equations governing stationary turbulence is obtained; these solutions, varying periodically in the vertical direction, enable a generalized Fourier analysis of the various correlation functions. According to these solutions, a Fourier analysis of correlations such as $\overline{u_\parallel(0)\, u'_\parallel(z)}$ of the vertical velocities at two points directly above one another and separated by a distance z, cannot include wave-lengths less than a certain minimum value depending on the physical parameters and on the temperature gradient maintained. We may thus speak of a smallest size for the eddies. Further, it appears that the field of turbulence can be analyzed into two modes characterized by the kinetic energy being confined, principally, to the vertical or to the horizontal direction.

1. Introduction

In nature the most common cause for the occurrence of turbulence is thermal instability which results from the joint effects of gravity and a superadiabatic temperature gradient. When turbulence occurs under these conditions the situation generally is that the amount of heat that has to be transported requires the surpassing of the adiabatic temperature gradient. Thus if $-|\mathrm{d}T/\mathrm{d}z|$ is the temperature gradient that is present in a gaseous plane-parallel atmosphere, the condition that the adiabatic gradient be exceeded is (cf. Brunt 1939)

$$\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right| > \frac{g\alpha T}{c_p}, \tag{1}$$

where g denotes the value of gravity, α the coefficient of volume expansion, c_p the specific heat at constant pressure and T the temperature on the absolute scale. The understanding of turbulence initiated by such thermal instability is clearly a matter of some importance. It is the object of this paper to examine how far the methods currently employed in the statistical theories of turbulence can be extended to the treatment of this problem. The problem has an added interest in that in treating it we must explicitly take into account the source from which turbulence derives its energy: an aspect of the general turbulence problem which has not, so far, received much attention.

It is apparent that a full discussion of turbulence caused by thermal instability allowing for compressibility, etc., will be a very difficult one. However, a theorem of Jeffreys (1930) enables us to treat the problem under somewhat simplified conditions: For, the theorem of Jeffreys states that when the density in the system does not vary greatly, the case of an incompressible fluid with the higher temperature on the under side is formally the same as a compressible fluid with a temperature gradient in excess of the adiabatic. Thus, an adverse temperature gradient (i.e., one which is negative if g acts in the direction of decreasing z) in an incompressible fluid is equivalent to an adiabatic excess $(\mathrm{d}T/\mathrm{d}z + g\alpha T/c_p)$ in a compressible fluid. Moreover, when treating an incompressible fluid in which the temperature is variable, we can often neglect the variations of density (caused by thermal expansion) except in so far as they modify gravity. Quite generally, the circumstances under which such a neglect is justified, have been discussed by Boussinesq (1903); and Rayleigh (1916) and Jeffreys (1926, 1928) have shown that it is justified in problems of the type we shall consider. Indeed, the particular problem we shall treat is closely related to the one investigated by Rayleigh and Jeffreys in their papers referred to, namely the stability of a layer of fluid heated below. (For a general account of these investigations, see Brunt (1939, chaps. XI and XII) and Wasiutynski (1946); the former discusses the problem from the meteorological and the latter from the astrophysical point of view.) Rayleigh, who initiated the study of this problem with a view to interpreting the experiments of Bénard (1900, 1901), showed that a liquid might be in stable equilibrium even if its density increases upwards (as would be the case when an adverse temperature gradient is present) provided its viscosity and heat conductivity are sufficiently high. More particularly, Rayleigh showed that a layer of liquid of height H with a free surface at both top and bottom with a temperature maintained constant over both will first become unstable when

$$\frac{g\alpha\,|\beta|}{\kappa\nu} H^4 = \frac{27}{4}\pi^4, \tag{2}$$

where $\beta = -|\beta|$ denotes the vertical temperature gradient and κ and ν are the coefficients of thermometric conductivity and kinematic viscosity, respectively. Rayleigh further showed that when the quantity on the left-hand side just exceeds $27\pi^4/4$ convection with a cellular pattern will set in; this is in qualitative agreement with the experiments of Bénard. In extending Rayleigh's discussion of the problem when one (or both) of the bounding surfaces is (are) not free Jeffreys (1926, 1928) derived a general differential equation governing the problem and showed how in these cases also the critical value of $g\alpha\,|\beta|\,H^4/\kappa\nu$ at which instability first sets in can be determined (in this connexion see also Low (1929)).

Experiments by Schmidt & Milverton (1935) carried out with the explicit purpose of putting the Rayleigh-Jeffreys criterion to a quantitative test have shown that for the predicted temperature gradient (for given values of the other parameters) instability does set in. Later experiments by Schmidt & Saunders (1938) have, however, indicated that while cellular convection of the type predicted (and first observed by Bénard) does set in when the temperature gradient given by the Rayleigh-Jeffreys criterion is exceeded, the convection changes from a 'cellular' to a truly 'turbulent' pattern for a much higher temperature gradient. Schmidt & Saunders themselves believed that the transition from 'cellular' to 'turbulent' convection is a sharp one; but an examination of their experimental points would not seem to support this latter conclusion as well established; this can be seen from plotting their experimental points in figure 2 without their 'guiding' lines. Also, there would appear to be theoretical difficulties in accepting the conclusion that the transition between the two types of convection is a sharp one (see § 11 below). In any case, it is clear that the condition for the occurrence of turbulence when an incompressible fluid is heated from below requires careful examination; it is in part the object of this paper.

2. The equations of Boussinesq; the differential equation of Jeffreys; and the criterion of Rayleigh

The equations of motion and heat conduction appropriate to the problem are

$$\rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial V}{\partial x_i} + \rho\nu\nabla^2 u_i \tag{3}$$

and

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \kappa \nabla^2 T, \tag{4}$$

where here and in the sequel summation over repeated indices is to be understood. In equation (3), V denotes the gravitational potential and the rest of the symbols have either been defined or have their usual meanings.

As we have already stated in § 1, we shall take into account the variation of density only in so far as it modifies the effect of the external field. Thus, in equation (3) we replace ρ which occurs in front of $\partial V/\partial x_i$ by

$$\rho = \rho_0(1 - \alpha\Delta T), \tag{5}$$

where α denotes the coefficient of volume expansion, ρ_0 the density corresponding to a certain mean temperature T_0 and ΔT is the deviation of the local temperature from T_0:

$$\Delta T = T - T_0; \tag{6}$$

and we regard ρ occurring elsewhere in equation (3) as a constant equal to ρ_0. On these assumptions equation (3) becomes

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho_0} - V\right) - \alpha\Delta T \frac{\partial V}{\partial x_i} + \nu\nabla^2 u_i. \tag{7}$$

With the variation of density due to thermal expansion allowed for in this manner we, from now on, treat u_i as a solenoidal vector:

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{8}$$

The approximations represented by equations (7) and (8) are those first introduced by Boussinesq (1903).

When the external field is that due to the action of gravity, we can write

$$\frac{\partial V}{\partial x_i} = -g\lambda_i \quad \text{and} \quad V = -g\lambda_j x_j, \tag{9}$$

where $\boldsymbol{\lambda}$ is a unit vector in the direction of the vertical. With the foregoing substitution, equation (7) becomes

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho_0} + g\lambda_j x_j\right) + \gamma\Delta T\lambda_i + \nu\nabla^2 u_i, \tag{10}$$

where we have written

$$\gamma = g\alpha. \tag{11}$$

Returning to the equation of heat conduction (4), we shall suppose that a constant mean temperature gradient $\beta = -|\beta|$ is maintained in the direction of $\boldsymbol{\lambda}$ by an external agency and that we can write

$$T = T_0 + \beta\lambda_j x_j + \theta, \quad \Delta T = \beta\lambda_j x_j + \theta, \tag{12}$$

where T_0 is a constant (cf. equation (6)),

$$\beta = \frac{\mathrm{d}\overline{T}}{\mathrm{d}x_j}\lambda_j, \tag{13}$$

and θ is the deviation of the temperature from its local mean value, $T_0 + \beta\lambda_j x_j$. Inserting (12) in equations (4) and (10), we have

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial \varpi}{\partial x_i} + \gamma\theta\lambda_i + \nu\nabla^2 u_i, \tag{14}$$

and

$$\frac{\partial \theta}{\partial t} + \beta\lambda_j u_j + u_j\frac{\partial \theta}{\partial x_j} = \kappa\nabla^2\theta, \tag{15}$$

where, for brevity, we have written

$$\varpi = \frac{p}{\rho_0} + g\lambda_j x_j - \tfrac{1}{2}\gamma\beta\lambda_i\lambda_j x_i x_j. \tag{16}$$

We shall now suppose that u_i and θ are small quantities of the first order and that we can ignore products and squares of them. On this approximation equations (14) and (15) reduce to

$$\frac{\partial u_i}{\partial t} = -\frac{\partial \varpi}{\partial x_i} + \gamma\theta\lambda_i + \nu\nabla^2 u_i \tag{17}$$

and

$$\frac{\partial \theta}{\partial t} = -\beta\lambda_j u_j + \kappa\nabla^2\theta. \tag{18}$$

Taking the divergence of equation (17) and remembering that u_i is solenoidal (cf. equation (8)) we find that

$$\nabla^2\varpi = \gamma\lambda_j\frac{\partial \theta}{\partial x_j}. \tag{19}$$

Equations (17) to (19) provide the basis for the Rayleigh-Jeffreys theory; and Jeffreys's differential equation can be derived from them in the following manner:

$$\begin{aligned}\nabla^2\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\theta &= -\nabla^2\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\beta\lambda_j u_j \\ &= \nabla^2\left[-\beta\lambda_j\left(-\frac{\partial \varpi}{\partial x_j} + \gamma\theta\lambda_j\right)\right] = \beta\lambda_j\frac{\partial}{\partial x_j}\nabla^2\varpi - \beta\gamma\nabla^2\theta \\ &= \beta\gamma\left(\lambda_i\lambda_j\frac{\partial^2\theta}{\partial x_i\partial x_j} - \nabla^2\theta\right).\end{aligned} \tag{20}$$

According to Rayleigh and Jeffreys the condition for marginal stability is obtained by setting $\partial/\partial t = 0$ in the equations of motion and heat conduction; in which case equation (20) reduces to

$$\nabla^6\theta = \frac{\beta\gamma}{\kappa\nu}\left(\lambda_i\lambda_j\frac{\partial^2\theta}{\partial x_i\partial x_j} - \nabla^2\theta\right). \tag{21}$$

If $\boldsymbol{\lambda}$ is chosen to be in the z-direction, $\boldsymbol{\lambda} = (0, 0, 1)$, and equation (21) becomes

$$\nabla^6\theta = -\frac{\beta\gamma}{\kappa\nu}\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right). \tag{22}$$

This is Jeffreys's differential equation (1926, equation (13)).

If in equation (22) we let $$\theta = Z(z)\sin\alpha_1 x\sin\alpha_2 y, \tag{23}$$

where α_1 and α_2 are constants, we find

$$\left(\frac{d^2}{dz^2}-\alpha^2\right)^3 Z = -\frac{|\beta|\gamma}{\kappa\nu}\alpha^2 Z, \tag{24}$$

where $$\alpha^2 = \alpha_1^2+\alpha_2^2. \tag{25}$$

Equation (24) admits a solution of the form

$$Z = Z_0 \sin az \tag{26}$$

where Z_0 is a constant, provided $$\frac{(a^2+\alpha^2)^3}{\alpha^2} = \frac{|\beta|\gamma}{\kappa\nu}. \tag{27}$$

For a given a the left-hand side has a minimum value which it attains for $\alpha^2 = \frac{1}{2}a^2$. Accordingly

$$\frac{|\beta|\gamma}{\kappa\nu} \geqslant \tfrac{27}{4}a^4. \tag{28}$$

Therefore, *for a given $|\beta|$ periodic fluctuations of temperature in the z-direction can exist only when the wave-length, λ, of the fluctuation is greater than a certain minimum wave-length, $\lambda_{\text{min.}}$, given by*

$$\lambda^4_{\text{min.}} = 108\pi^4\frac{\kappa\nu}{|\beta|\gamma}. \tag{29}$$

This is essentially Rayleigh's criterion (cf. equation (2)*).

3. Various correlations for describing turbulence caused by thermal instability

We shall consider a fluctuating (i.e. a turbulent) field of velocity (u_i) and temperature (θ) governed by equations (17) to (19). In order to describe such a field we shall introduce various correlations such as $\overline{\theta u_i'}$ and $\overline{u_i u_j'}$, between the simultaneous values of the field variables at two different points $P(x_i)$ and $P'(x_i')$ in the medium. In general, such correlations are vectors or tensors which require for their definitions a number of scalar functions. The number of such scalar functions required can be reduced considerably by assuming that the turbulent field is statistically homogeneous and further satisfies certain invariance properties for reflexion and rotation. For example, in homogeneous isotropic turbulence the number of scalars required for the definition of a general second-order tensor is two (cf. Robertson 1940); the number is further reduced to one if the tensor (like $\overline{u_i u_j'}$) is solenoidal in its indices. In discussing turbulence in the framework of equations (17) to (19), we clearly cannot assume that it is homogeneous and isotropic: the explicit appearance of a preferential direction ($\boldsymbol{\lambda}$) in the basic equations would already make this impossible. On

* The ratio 16 in the numerical factors of equations (2) and (29) arises from the fact that when a layer of liquid of height, H, is considered the lowest 'mode' (compatible with the top and bottom surfaces being free) is obtained when the fluctuation in the z-direction is given by $\sin(\pi z/H)$, i.e. when $a = \pi/H$; on the other hand, we have set $a = 2\pi/\lambda$ as the definition of the wave-length.

the other hand, the assumption of axisymmetry in the sense described by Batchelor (1946) and Chandrasekhar (1950*a*, referred to hereafter as A.T.) would be *compatible* with the equations though it cannot be concluded on that account that the assumption of axisymmetry is necessarily justified. Indeed, one can object to regarding the various correlations as axisymmetric vectors and tensors (when they are not scalars) with representations as given in A.T., on the ground that the assumption of *homogeneity* is involved; for a finite layer of fluid the assumption of strict homogeneity clearly cannot be defended. But if we restrict ourselves to regions in the fluid which are far from the bounding surfaces, we may expect that the assumption of axisymmetric turbulence may be approximately realized. This assumption will be made in this paper.

We shall assume then, that the various correlations which we shall find it necessary to introduce are, when they are not scalars, axisymmetric vectors and tensors in the strict sense defined in A.T. Correlations in which one of the field variables is a velocity component, u_i, will be solenoidal in that index; the corresponding vectors and tensors can then be defined uniquely in a gauge-invariant way in terms of certain defining scalars as described in A.T., §§ 3 to 5. Thus, an axisymmetric vector, L_i, solenoidal in i can be expressed in terms of a single defining scalar, L, which is a function only of r (the distance between the two points considered) and μ (the direction cosine of the angle between the directions $\boldsymbol{\xi}$ ($\xi_i = x_i'-x_i$) and $\boldsymbol{\lambda}$) in the form (A.T., equation (26)),

$$L_i = -(r\mu D_r + D_\mu) L\xi_i + (r^2 D_r + r\mu D_\mu + 2) L\lambda_i, \tag{30}$$

where D_r and D_μ are the differential operators (A.T., equations (6))

$$D_r = \frac{1}{r}\frac{\partial}{\partial r} - \frac{\mu}{r^2}\frac{\partial}{\partial \mu} \quad \text{and} \quad D_\mu = \frac{1}{r}\frac{\partial}{\partial \mu}. \tag{31}$$

Similarly, a tensor like $\overline{u_i u_j'}$ which is symmetrical and solenoidal in its indices can be defined in terms of two scalars Q_1 and Q_2; the explicit representation is given in A.T., equations (48) to (50).

The various correlations which we shall find it necessary to introduce are listed in table 1 together with their defining scalars.

TABLE 1. VARIOUS CORRELATIONS AND THEIR DEFINING SCALARS

correlation	defining scalar	remarks
$\overline{\theta\theta'} = \Theta$	Θ	an even function of r and μ
$\overline{\theta u_i'} = L_i^{(1)}$; $\overline{\theta' u_i} = L_i^{(2)}$	l_1, l_2	from homogeneity, $L_i^{(1)}(\boldsymbol{\lambda}) = -L_i^{(2)}(-\boldsymbol{\lambda})$; $l_1(r,\mu) = l_2(r,-\mu)$
$\frac{1}{2}(\overline{\theta u_i'} + \overline{\theta' u_i}) = \Lambda_i$	Λ	$\Lambda = \frac{1}{2}(l_1+l_2)$; an even function of r and μ
$\frac{1}{2}(\overline{\theta u_i'} - \overline{\theta' u_i}) = L_i$	L	$L = \frac{1}{2}(l_1-l_2)$; an odd function of r and μ
$\frac{1}{2}(\overline{\theta'\varpi} - \overline{\theta\varpi'}) = \Phi$	Φ	Φ an odd function of r and μ
$\frac{1}{2}(\overline{\theta'\varpi} + \overline{\theta\varpi'}) = \Psi$	Ψ	Ψ an even function of r and μ; actually $\Psi \equiv 0$ (cf. equation (48))
$\overline{\varpi u_i'} = P_i^{(1)}$; $\overline{\varpi' u_i} = P_i^{(2)}$	p_1, p_2	from homogeneity, $P_i^{(1)}(\boldsymbol{\lambda}) = -P_i^{(2)}(-\boldsymbol{\lambda})$; $p_1(r,\mu) = p_2(r,-\mu)$
$\frac{1}{2}(\overline{\varpi u_i'} - \overline{\varpi' u_i}) = \Pi_i$	Π	$\Pi = \frac{1}{2}(p_1-p_2)$; an odd function of r and μ
$\frac{1}{2}(\overline{\varpi u_i'} + \overline{\varpi' u_i}) = P_i$	P	$P = \frac{1}{2}(p_1+p_2)$; an even function of r and μ
$\overline{u_i u_j'} = Q_{ij}$	Q_1, Q_2	Q_1 and Q_2 even functions of r and μ; for explicit representation of tensor, see A.T., equations (48) to (50)

4. Equations governing the scalars

We shall now derive the equation governing the various scalars introduced in the preceding section.

Multiplying equation (18) by θ' (the value of θ at x_i') and averaging we obtain

$$\overline{\theta' \frac{\partial \theta}{\partial t}} = -\beta \lambda_j \overline{\theta' u_j} + \kappa \nabla^2 \overline{\theta \theta'}. \tag{32}$$

Interchanging the primed and the unprimed variables in this equation, we have

$$\overline{\theta \frac{\partial \theta'}{\partial t}} = -\beta \lambda_j \overline{\theta u_j'} + \kappa \nabla^2 \overline{\theta \theta'}. \tag{33}$$

Now adding equations (32) and (33), we obtain (cf. table 1)

$$\frac{\partial \Theta}{\partial t} = -2\beta \lambda_j \Lambda_j + 2\kappa \Delta_3 \Theta, \tag{34}$$

where (cf. A.T., equation (31))

$$\begin{aligned}\Delta_3 &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2} - \frac{2\mu}{r^2}\frac{\partial}{\partial \mu} \\ &= r^2 D_{rr} + 2r\mu D_{r\mu} + D_{\mu\mu} + 3D_r\end{aligned} \tag{35}$$

is the operator defining the axially symmetric wave equation in three dimensions; quite generally, the corresponding operator in n dimensions is

$$\begin{aligned}\Delta_n &= \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2} - \frac{n-1}{r^2}\mu\frac{\partial}{\partial \mu} \\ &= r^2 D_{rr} + 2r\mu D_{r\mu} + D_{\mu\mu} + nD_r.\end{aligned} \tag{36}$$

From the expression for Λ_j in terms of its defining scalar Λ (cf. equation (30)) we find

$$\begin{aligned}\lambda_j \Lambda_j &= -\lambda_j \xi_j (r\mu D_r + D_\mu)\,\Lambda + \lambda_j \lambda_j (r^2 D_r + r\mu D_\mu + 2)\,\Lambda \\ &= [r^2(1-\mu^2)\,D_r + 2]\,\Lambda.\end{aligned} \tag{37}$$

Thus, equation (34) becomes

$$\frac{\partial \Theta}{\partial t} = -2\beta[r^2(1-\mu^2)\,D_r + 2]\,\Lambda + 2\kappa \Delta_3 \Theta. \tag{38}$$

Next multiplying equation (17) by θ' and averaging we obtain

$$\overline{\theta' \frac{\partial u_i}{\partial t}} = -\frac{\partial}{\partial x_i}\overline{\varpi \theta'} + \gamma \lambda_i \overline{\theta \theta'} + \nu \nabla^2 \overline{\theta' u_i}. \tag{39}$$

Similarly, considering equation (18) in the primed variables, multiplying by u_i and averaging, we obtain

$$\overline{u_i \frac{\partial \theta'}{\partial t}} = -\beta \lambda_j \overline{u_i u_j'} + \kappa \nabla^2 \overline{\theta' u_i}. \tag{40}$$

Adding equations (39) and (40) and remembering that $\partial/\partial x_i = -\partial/\partial \xi_i$, we obtain

$$\frac{\partial}{\partial t}\overline{\theta' u_i} = +\frac{\partial}{\partial \xi_i}\overline{\varpi \theta'} + \gamma \lambda_i \Theta - \beta \lambda_j Q_{ij} + (\kappa + \nu)\,\nabla^2 \overline{\theta' u_i}. \tag{41}$$

Interchanging the primed and the unprimed quantities in this equation, we have

$$\frac{\partial}{\partial t}\overline{\theta u_i'} = -\frac{\partial}{\partial \xi_i}\overline{\varpi'\theta}+\gamma\lambda_i\Theta-\beta\lambda_j Q_{ij}+(\kappa+\nu)\nabla^2\overline{\theta u_i'}, \tag{42}$$

where we have made use of the symmetry of Q_{ij} in its indices. Now adding and subtracting equations (41) and (42), we obtain (cf. table 1)

$$\frac{\partial \Lambda_i}{\partial t} = \left(\frac{\partial \Phi}{\partial \xi_i}+\gamma\lambda_i\Theta\right)-\beta\lambda_j Q_{ij}+(\kappa+\nu)\nabla^2\Lambda_i \tag{43}$$

and

$$\frac{\partial L_i}{\partial t} = \frac{\partial \Psi}{\partial \xi_i}+(\kappa+\nu)\nabla^2 L_i. \tag{44}$$

The equations governing Φ and Ψ which occur in equations (43) and (44) can be obtained by multiplying equation (19) in the primed (or the unprimed) variables and multiplying by θ (or θ') and averaging; thus

$$\Delta_3\overline{\varpi\theta'} = -\gamma\lambda_j\frac{\partial\Theta}{\partial\xi_j} \quad \text{and} \quad \Delta_3\overline{\varpi'\theta} = +\gamma\lambda_j\frac{\partial\Theta}{\partial\xi_j}. \tag{45}$$

Adding and subtracting these equations we obtain

$$\Delta_3\Psi = 0 \tag{46}$$

and

$$\Delta_3\Phi = -\gamma\lambda_j\frac{\partial\Theta}{\partial\xi_j} = -\gamma(r\mu D_r+D_\mu)\,\Theta. \tag{47}$$

Equation (46) does not allow any non-trivial solution which is bounded everywhere and vanishes at infinity. Accordingly, we shall write

$$\Psi \equiv 0. \tag{48}*$$

Equation (44) now becomes

$$\frac{\partial L_i}{\partial t} = (\kappa+\nu)\nabla^2 L_i. \tag{49}$$

The defining scalars of the solenoidal axisymmetric vectors which occur on either side of this equation are (cf. A.T., equations (29) to (31)):

$$\frac{\partial L}{\partial t} \quad \text{and} \quad \Delta_5 L. \tag{50}$$

Equation (49) is therefore equivalent to the scalar equation,

$$\frac{\partial L}{\partial t} = (\kappa+\nu)\,\Delta_5 L. \tag{51}$$

This is the equation of heat conduction for an axially symmetric distribution of matter in five-dimensional space (cf. Chandrasekhar 1950*b*).

Before we can write the equivalent scalar equation of (43), we must find the defining scalars of the terms besides $\nabla^2\Lambda_i$ on the right-hand side of this equation. From the explicit expression for Q_{ij} given in A.T., equations (48) to (50), we find on contraction that

$$\lambda_j Q_{ij} = +(r\mu D_r+D_\mu)\,Q_1\xi_i-(r^2D_r+r\mu D_\mu+2)\,Q_1\lambda_i. \tag{52}$$

* We can conclude this also from the solenoidal character of $\partial\Psi/\partial\xi_i$ which is required by equation (44).

The defining scalar of $\lambda_j Q_{ij}$ is, therefore, $-Q_1$. Considering next the term $(\partial\Phi/\partial\xi_i+\gamma\lambda_i\Theta)$ we first observe that this must be solenoidal in i (since all the other terms in equation (43) are solenoidal in i) and, therefore, must be expressible in terms of a defining scalar X in the form

$$\frac{\partial\Phi}{\partial\xi_i}+\gamma\lambda_i\Theta = -(r\mu D_r+D_\mu)\,X\xi_i+(r^2D_r+r\mu D_\mu+2)\,X\lambda_i. \tag{53}$$

On the other hand,

$$\frac{\partial\Phi}{\partial\xi_i}+\gamma\lambda_i\Theta = D_r\Phi\xi_i+(D_\mu\Phi+\gamma\Theta)\,\lambda_i. \tag{54}$$

From equations (53) and (54) we conclude that

$$D_r\Phi = -(r\mu D_r+D_\mu)\,X \tag{55}$$

and

$$D_\mu\Phi+\gamma\Theta = +(r^2D_r+r\mu D_\mu+2)\,X. \tag{56}$$

It can be readily verified that equations (55) and (56) represent the general solution (bounded everywhere and vanishing at infinity) of equation (47). Thus, by using equations (55) and (56) in conjunction with the identity established in A.T., equation (14), we obtain

$$\begin{aligned}(r\mu D_r+D_\mu)\,(D_\mu\Phi+\gamma\Theta) &= (r\mu D_r+D_\mu)\,(r^2D_r+r\mu D_\mu+2)\,X\\ &= (r^2D_r+r\mu D_\mu+3)\,(r\mu D_r+D_\mu)\,X,\end{aligned} \tag{57}$$

or

$$(r\mu D_r+D_\mu)\,(D_\mu\Phi+\gamma\Theta) = -(r^2D_r+r\mu D_\mu+3)\,D_r\Phi. \tag{58}$$

Rearranging this last equation, we recover (47).

Returning to equation (43), we can now write down the defining scalars of the various axisymmetric solenoidal vectors which occur in this equation. They are

$$\frac{\partial\Lambda}{\partial t},\quad X,\quad -Q_1\quad\text{and}\quad \Delta_5\Lambda. \tag{59}$$

Equation (43) is therefore equivalent to

$$\frac{\partial\Lambda}{\partial t} = X+\beta Q_1+(\kappa+\nu)\,\Delta_5\Lambda, \tag{60}$$

where X is defined implicitly through equations (55) and (56).

An equation directly relating the scalars X and Θ can be obtained as follows: Operating equations (55) and (56) by D_μ and D_r, respectively, we have

$$D_{r\mu}\Phi = -D_\mu(r\mu D_r+D_\mu)\,X = -(r\mu D_{r\mu}+D_{\mu\mu}+D_r)\,X \tag{61}$$

and

$$\begin{aligned}D_{r\mu}\Phi+\gamma D_r\Theta &= D_r(r^2D_r+r\mu D_\mu+2)\,X\\ &= (r^2D_{rr}+r\mu D_{r\mu}+4D_r)\,X;\end{aligned} \tag{62}$$

and eliminating $D_{r\mu}\Phi$ from these equations, we obtain

$$\Delta_5 X = \gamma D_r\Theta. \tag{63}$$

Finally, we can derive an equation governing the rate of change of Q_{ij} by treating equation (17) in the usual fashion by multiplying by u_j', averaging the resulting equation, etc. In this manner we find

$$\frac{\partial Q_{ij}}{\partial t} = S_{ij}+2\nu\nabla^2 Q_{ij}, \tag{64}$$

where

$$S_{ij} = \left(\frac{\partial}{\partial\xi_i}\overline{\varpi u_j'}+\gamma\lambda_i\overline{\theta u_j'}\right)-\left(\frac{\partial}{\partial\xi_j}\overline{\varpi' u_i}-\gamma\lambda_j\overline{\theta' u_i}\right). \tag{65}$$

From equation (64) it is evident that since Q_{ij} and $\nabla^2 Q_{ij}$ are both symmetrical in their indices and solenoidal, the tensor S_{ij} must also be symmetrical in its indices and solenoidal; S_{ij} must therefore be definable in terms of two scalars, say, S_1 and S_2.

If Q_1 and Q_2 are the defining scalars of Q_{ij} (cf. table 1) then those of $\nabla^2 Q_{ij}$ are (cf. A.T., equation (61))

$$\Delta_5 Q_1 \quad \text{and} \quad \Delta_5 Q_2 + 2D_{\mu\mu} Q_1. \tag{66}$$

The equivalent scalar equations of (64) are, therefore,

$$\frac{\partial Q_1}{\partial t} = S_1 + 2\nu \Delta_5 Q_1 \tag{67}$$

and

$$\frac{\partial Q_2}{\partial t} = S_2 + 2\nu(\Delta_5 Q_2 + 2D_{\mu\mu} Q_1). \tag{68}$$

It remains to express S_1 and S_2 in terms of the other scalars of the problem. For this purpose consider first the tensor (cf. table 1)

$$s_{ij} = \frac{\partial}{\partial \xi_i} \overline{\varpi u_j'} + \gamma \lambda_i \overline{\theta u_j'} = \frac{\partial P_j^{(1)}}{\partial \xi_i} + \gamma \lambda_i L_j^{(1)}. \tag{69}$$

Now $P_j^{(1)}$ is solenoidal in j with a defining scalar p_1 (cf. table 1) and according to A.T. equation (77) the defining scalars of $\partial P_j^{(1)}/\partial \xi_i$ are

$$-(r\mu D_r + D_\mu)\, p_1, \quad D_\mu(r^2 D_r + r\mu D_\mu + 2)\, p_1 \quad \text{and} \quad D_r(r^2 D_r + r\mu D_\mu + 2)\, p_1. \tag{70}$$

To determine the defining scalars of $\lambda_i L_j^{(1)}$ we proceed as follows: First we verify that

$$\lambda_i L_j^{(1)} = -\lambda_i \xi_j (r\mu D_r + D_\mu)\, l_1 + \lambda_i \lambda_j (r^2 D_r + r\mu D_\mu + 2)\, l_1 \tag{71}$$

is the curl with respect to the second index of the skew tensor (cf. A.T., equation (39))

$$l_1 \lambda_i \epsilon_{jlm} \lambda_l \xi_m = l_1 (\lambda_j \epsilon_{ilm} \lambda_l \xi_m + r\mu \epsilon_{ijk} \lambda_k - \epsilon_{ijk} \xi_k). \tag{72}$$

In accordance with A.T., equation (40), we must replace $l_1 r\mu \epsilon_{ijk} \lambda_k$ on the right-hand side of (72) by

$$D_r(r\mu l_1)\, \xi_j \epsilon_{ilm} \lambda_l \xi_m + D_\mu(r\mu l_1)\, \lambda_j \epsilon_{ilm} \lambda_l \xi_m. \tag{73}$$

The tensor $\lambda_i L_j^{(1)}$ is therefore the curl with respect to the second index of the skew tensor

$$-l_1 \epsilon_{ijk} \xi_k + D_r(r\mu l_1)\, \xi_j \epsilon_{ilm} \lambda_l \xi_m + [l_1 + D_\mu(r\mu l_1)]\, \lambda_j \epsilon_{ilm} \lambda_l \xi_m; \tag{74}$$

its defining scalars are, therefore,

$$-l_1, \quad D_\mu(r\mu l_1) + l_1 \quad \text{and} \quad D_r(r\mu l_1). \tag{75}$$

Combining (70) and (75), we observe that the defining scalars of s_{ij} are

$$-(r\mu D_r + D_\mu)\, p_1 - \gamma l_1, \quad D_\mu(r^2 D_r + r\mu D_\mu + 2)\, p_1 + \gamma l_1 + \gamma D_\mu(r\mu l_1)$$

and

$$D_r(r^2 D_r + r\mu D_\mu + 2)\, p_1 + \gamma D_r(r\mu l_1). \tag{76}$$

Now from the properties (cf. table 1)

$$\overline{\varpi u_j'}(r, +\mu) = -\overline{\varpi' u_j}(r, -\mu) \quad \text{and} \quad \overline{\theta u_j'}(r, +\mu) = -\overline{\theta' u_j}(r, -\mu), \tag{77}$$

it follows that

$$S_{ij} = s_{ij}(+\boldsymbol{\lambda}) + s_{ji}(-\boldsymbol{\lambda}). \tag{78}$$

From the symmetry and the solenoidal character of S_{ij} in its indices we can infer that the same must be true of s_{ij} (cf. remarks following A.T., equation (144)); and this requires that among the defining scalars of s_{ij} the following relation holds (A.T., equation (47)):

$$D_r(r^2D_r+r\mu D_\mu+2)\,p_1+\gamma D_r(r\mu l_1) = -D_\mu[(r\mu D_r+D_\mu)\,p_1+\gamma l_1]. \quad (79)$$

On further simplification the foregoing equation reduces to

$$\Delta_5 p_1 = -\gamma(r\mu D_r+D_\mu)\,l_1. \quad (80)$$

This last relation between p_1 and l_1 also follows from writing, in terms of the defining scalars, the equation

$$\nabla^2 P_i^{(1)} = -\gamma\lambda_j \frac{\partial L_i^{(1)}}{\partial \xi_j}, \quad (81)$$

which one obtains from equation (19) after multiplying by u_i' and averaging. The two defining scalars of s_{ij} are, therefore,

$$-(r\mu D_r+D_\mu)\,p_1-\gamma l_1 \quad \text{and} \quad D_\mu(r^2D_r+r\mu D_\mu+2)\,p_1+\gamma l_1+\gamma D_\mu(r\mu l_1). \quad (82)$$

Returning to equation (65), and considering the tensor,

$$\frac{\partial}{\partial \xi_j}\overline{\varpi' u_i}-\gamma\lambda_j\overline{\theta' u_i} = \frac{\partial P_i^{(2)}}{\partial \xi_j}-\gamma\lambda_j L_i^{(2)}, \quad (83)$$

and reasoning as in the case of s_{ij} (though in this case some additional care is necessary since the tensor (83), in contrast to (69), is, per definition, solenoidal in its *first* index) we find that its defining scalars are:

$$-(r\mu D_r+D_\mu)\,p_2+\gamma l_2 \quad \text{and} \quad D_\mu(r^2D_r+r\mu D_\mu+2)\,p_2-\gamma l_2-\gamma D_\mu(r\mu l_2). \quad (84)$$

Finally, combining (82) and (84), we obtain for S_{ij} the defining scalars

$$S_1 = -2(r\mu D_r+D_\mu)\,\Pi-2\gamma\Lambda \quad (85)$$

and

$$S_2 = 2D_\mu(r^2D_r+r\mu D_\mu+2)\,\Pi+2\gamma\Lambda+2\gamma D_\mu(r\mu\Lambda). \quad (86)$$

The explicit form of the equations governing Q_1 and Q_2 are, therefore,

$$\frac{\partial Q_1}{\partial t} = -2(r\mu D_r+D_\mu)\,\Pi-2\gamma\Lambda+2\nu\Delta_5 Q_1 \quad (87)$$

and

$$\frac{\partial Q_2}{\partial t} = 2D_\mu(r^2D_r+r\mu D_\mu+2)\,\Pi+2\gamma\Lambda+2\gamma D_\mu(r\mu\Lambda)+2\nu(\Delta_5 Q_2+2D_{\mu\mu}Q_1). \quad (88)$$

Equations governing Π and P can be obtained by combining equation (81) with the similar one,

$$\nabla^2 P_i^{(2)} = +\gamma\lambda_j \frac{\partial L_i^{(2)}}{\partial \xi_j}, \quad (89)$$

which one obtains by multiplying equation (19) in the primed variables by u_i and averaging. Thus, we find

$$\nabla^2\Pi_i = -\gamma\lambda_j\frac{\partial \Lambda_i}{\partial \xi_j} \quad \text{and} \quad \nabla^2 P_i = -\gamma\lambda_j\frac{\partial L_i}{\partial \xi_j}. \quad (90)$$

In terms of defining scalars, the foregoing equations are equivalent to (cf. equation (80))

$$\Delta_5\,\Pi = -\gamma(r\mu D_r+D_\mu)\,\Lambda \quad \text{and} \quad \Delta_5 P = -\gamma(r\mu D_r+D_\mu)\,L. \quad (91)$$

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It can be verified that for bounded functions Π and Λ vanishing at infinity, the solution of the equation for Π can be expressed in terms of an arbitrary (bounded) function Y in the form (cf. equations (47), (55) and (56))

$$D_r \Pi = -(r\mu D_r + D_\mu)\, Y \tag{92}$$

and

$$D_\mu \Pi + \gamma\Lambda = +(r^2 D_r + r\mu D_\mu + 4)\, Y. \tag{93}$$

And eliminating Π between these two last equations by reductions analogous to those employed with respect to equations (55) and (56) (cf. equations (61) to (63)) we obtain

$$\Delta_7 Y = \gamma D_r \Lambda. \tag{94}$$

The equation for P can be similarly reduced.

Using equations (92) and (93) we can simplify the right-hand side of equation (87). Thus,

$$\begin{aligned}(r\mu D_r + D_\mu)\,\Pi + \gamma\Lambda &= r\mu D_r \Pi + (D_\mu \Pi + \gamma\Lambda)\\ &= -r\mu(r\mu D_r + D_\mu)\, Y + (r^2 D_r + r\mu D_\mu + 4)\, Y\\ &= [r^2(1-\mu^2)\, D_r + 4]\, Y,\end{aligned} \tag{95}$$

and we can write

$$\frac{\partial Q_1}{\partial t} = -2[r^2(1-\mu^2)\, D_r + 4]\, Y + 2\nu\Delta_5 Q_1. \tag{96}$$

Collecting the various equations, we have the following set of equations governing the scalars defined in table 1:

$$\frac{\partial \Theta}{\partial t} = 2\,|\beta|\,[r^2(1-\mu^2)\, D_r + 2]\,\Lambda + 2\kappa\Delta_3 \Theta, \tag{97}$$

$$\frac{\partial \Lambda}{\partial t} = X - |\beta|\, Q_1 + (\kappa+\nu)\,\Delta_5 \Lambda, \tag{98}$$

$$\frac{\partial Q_1}{\partial t} = -2[r^2(1-\mu^2)\, D_r + 4]\, Y + 2\nu\Delta_5 Q_1, \tag{99}$$

$$\frac{\partial Q_2}{\partial t} = 2D_\mu[(r^2 D_r + 2)\,\Pi + r\mu(D_\mu \Pi + \gamma\Lambda)] + 2\gamma\Lambda + 2\nu(\Delta_5 Q_2 + 2D_{\mu\mu} Q_1), \tag{100}$$

$$D_r \Phi = -(r\mu D_r + D_\mu)\, X, \quad D_\mu \Phi + \gamma\Theta = (r^2 D_r + r\mu D_\mu + 2)\, X, \tag{101}$$

$$D_r \Pi = -(r\mu D_r + D_\mu)\, Y, \quad D_\mu \Pi + \gamma\Lambda = (r^2 D_r + r\mu D_\mu + 4)\, Y, \tag{102}$$

$$\Delta_5 X = \gamma D_r \Theta, \quad \Delta_7 Y = \gamma D_r \Lambda, \tag{103}$$

$$\frac{\partial L}{\partial t} = (\kappa+\nu)\,\Delta_5 L \quad \text{and} \quad \Delta_5 P = -\gamma(r\mu D_r + D_\mu)\, L. \tag{104}$$

5. The rates of change of the mean squares of the velocity components and the temperature fluctuations

The scalars Q_1, Q_2, Λ, P, X and Y are even functions of r and μ while L, Φ and Π are odd functions of r and μ (cf. table 1). For small values of r we can therefore assume series expansions of the forms:

$$\left.\begin{aligned} Q_1 &= \alpha_{00} + r^2(\alpha_{02} + \alpha_{22}\mu^2) + \dots, & Q_2 &= \beta_{00} + r^2(\beta_{02} + \beta_{22}\mu^2) + \dots,\\ \Theta &= \theta_{00} + r^2(\theta_{02} + \theta_{22}\mu^2) + \dots, & \Lambda &= \lambda_{00} + r^2(\lambda_{02} + \lambda_{22}\mu^2) + \dots,\\ X &= x_{00} + r^2(x_{02} + x_{22}\mu^2) + \dots, & Y &= y_{00} + r^2(y_{02} + y_{22}\mu^2) + \dots,\\ \Pi &= r\mu\varpi_{00} + r^3\mu(\varpi_{02} + \varpi_{22}\mu^2) + \dots, & \Phi &= r\mu\phi_{00} + r^3\mu(\phi_{02} + \phi_{22}\mu^2) + \dots \end{aligned}\right\} \tag{105}$$

The correlation of the various field variables, with one another, at the same point and their rates of change are related to the coefficients in the foregoing series expansions. Thus, the mean square velocities $\overline{u_\parallel^2}$ and $\overline{u_\perp^2}$ parallel and perpendicular, respectively, to the direction $\boldsymbol{\lambda}$ are given by (cf. A.T. equations (108))

$$\overline{u_\parallel^2} = -2\alpha_{00} \quad \text{and} \quad \overline{u_\perp^2} = -(2\alpha_{00}+\beta_{00}); \tag{106}$$

and the mean square velocity $\overline{u^2}$ is given by

$$\overline{u^2} = \overline{u_\parallel^2}+2\overline{u_\perp^2} = -2(3\alpha_{00}+\beta_{00}). \tag{107}$$

Also, from the definition of $\Theta(r,\mu)$ as $\overline{\theta\theta'}$ it follows that

$$\overline{\theta^2} = \theta_{00}. \tag{108}$$

The meaning of the coefficient λ_{00} in the expansion of Λ is also of some interest. By definition

$$\tfrac{1}{2}(\overline{\theta u_i'}+\overline{\theta' u_i}) = -(r\mu D_r+D_\mu)\,\Lambda\xi_i+(r^2D_r+r\mu D_\mu+2)\,\Lambda\lambda_i. \tag{109}$$

For $u_i = u_\parallel$ in the direction of $\boldsymbol{\lambda}$, $\lambda_i = 1$ and we have

$$\tfrac{1}{2}(\overline{\theta u_\parallel'}+\overline{\theta' u_\parallel}) = -(r\mu D_r+D_\mu)\,\Lambda\xi_\parallel+(r^2D_r+r\mu D_\mu+2)\,\Lambda. \tag{110}$$

Letting $r\to 0$ in equation (110), we find

$$\overline{\theta u_\parallel} = 2\lambda_{00}. \tag{111}$$

An alternative form of this last equation is

$$2\gamma\lambda_{00} = g\alpha\overline{\theta u_\parallel} = -\frac{g}{\rho_0}\overline{\delta\rho u_\parallel}, \tag{112}$$

where $\delta\rho$ denotes the fluctuation of density from the local mean value. Since a positive $\delta\rho$ at a point implies that the material in the neighbourhood will have a tendency to sink and a negative $\delta\rho$ implies a corresponding tendency to rise, there will be a negative correlation between $\delta\rho$ and $u_\parallel$; $\overline{\delta\rho u_\parallel}$ will, therefore, be negative and λ_{00} will be positive. Moreover, it is clear that $2\gamma\lambda_{00}$ *represents the rate of liberation of potential energy, per unit mass, by the action of gravity.*

The equations governing the rates of change of the mean square velocity components and temperature fluctuations can be obtained by inserting the expansions (105) in equations (97) to (102). In this manner we obtain

$$\left.\begin{aligned} x_{02}+x_{22} &= -\phi_{02}, & y_{02}+y_{22} &= -\varpi_{02},\\ \phi_{00}+\gamma\theta_{00} &= 2x_{00}, & \varpi_{00}+\gamma\lambda_{00} &= 4y_{00},\\ \phi_{02}+\gamma\theta_{02} &= 4x_{02}, & \varpi_{02}+\gamma\lambda_{02} &= 6y_{02},\\ 3\phi_{22}+\gamma\theta_{22} &= 4x_{22}, & 3\varpi_{22}+\gamma\lambda_{22} &= 6y_{22}, \end{aligned}\right\} \tag{113}$$

and

$$\frac{\mathrm{d}\theta_{00}}{\mathrm{d}t} = 4\,|\beta|\,\lambda_{00}+2\kappa(6\theta_{02}+2\theta_{22}), \tag{114}$$

$$\frac{\mathrm{d}\lambda_{00}}{\mathrm{d}t} = x_{00}-|\beta|\,\alpha_{00}+(\kappa+\nu)\,(10\lambda_{02}+2\lambda_{22}), \tag{115}$$

$$\frac{\mathrm{d}\alpha_{00}}{\mathrm{d}t} = -8y_{00}+2\nu(10\alpha_{02}+2\alpha_{22}), \tag{116}$$

$$\frac{\mathrm{d}\beta_{00}}{\mathrm{d}t} = 6\varpi_{00}+4\gamma\lambda_{00}+2\nu(10\beta_{02}+2\beta_{22}+4\alpha_{22}). \tag{117}$$

In terms of $\overline{u_\parallel^2}$ and $\overline{u_\perp^2}$ equations (116) and (117) take the forms

$$\frac{1}{2}\frac{d\overline{u_\parallel^2}}{dt} = 2\gamma\lambda_{00} + 2\varpi_{00} - 2\nu(10\alpha_{02} + 2\alpha_{22}) \tag{118}$$

and

$$\frac{d}{dt}(\overline{u_\parallel^2} - \overline{u_\perp^2}) = 6\varpi_{00} + 4\gamma\lambda_{00} + 2\nu(10\beta_{02} + 2\beta_{22} + 4\alpha_{22}), \tag{119}$$

where in equation (118) we have replaced $8y_{00}$ by $2\gamma\lambda_{00} + 2\varpi_{00}$ in accordance with (113). Eliminating $d\overline{u_\parallel^2}/dt$ from equation (119) we find

$$\frac{d\overline{u_\perp^2}}{dt} = -2\varpi_{00} - 4\nu(10\alpha_{02} + 4\alpha_{22} + 5\beta_{02} + \beta_{22}). \tag{120}$$

Substituting for $2\gamma\lambda_{00}$ from equation (112) we can rewrite equation (118) in the form

$$\frac{1}{2}\frac{d\overline{u_\parallel^2}}{dt} = -\frac{g}{\rho_0}\overline{\delta\rho u_\parallel} + 2\varpi_{00} - 2\nu(10\alpha_{02} + 2\alpha_{22}). \tag{121}$$

Combining equations (120) and (121) we obtain the equation governing the rate of dissipation of kinetic energy:

$$\frac{1}{2}\frac{d\overline{u^2}}{dt} = \frac{1}{2}\frac{d}{dt}(\overline{u_\parallel^2} + 2\overline{u_\perp^2}) = -\frac{g}{\rho_0}\overline{\delta\rho u_\parallel} - 4\nu(15\alpha_{02} + 5\alpha_{22} + 5\beta_{02} + \beta_{22}). \tag{122}$$

Comparing equations (120) to (122) with A.T. equations (125) and (126) we observe that the terms proportional to ν are exactly the same in these equations. This agreement (which is clearly necessary) enables us to interpret equations (120) to (122) in the following manner: The rate of change of the mean square kinetic energy in the direction parallel to $\boldsymbol{\lambda}$ is the net result of the increase consequent to the release of potential energy by the action of gravity, the decrease consequent to the transfer of energy from this direction to the perpendicular direction by the action of the pressure term in the equation of motion and the decrease consequent to the dissipation of the energy by the action of viscosity. There is no net gain in the kinetic energy perpendicular to $\boldsymbol{\lambda}$ by the action of gravity; whatever gain there is, is due to the transfer of energy from the parallel component by the action of pressure. And finally, the rate of increase of the total kinetic energy is entirely due to the balance between the energy released by the work done by gravity and the energy lost by viscous dissipation. It would appear that equations (120) to (122) express results which have been derived differently by Richardson (1920) in his considerations relating to the supply of energy to atmospheric eddies.

Turning next to equation (114), we can rewrite it in the form (cf. equation (108))

$$\frac{d\overline{\theta^2}}{dt} = 2\,|\beta|\,\overline{\theta u_\parallel} + 2\kappa(6\theta_{02} + 2\theta_{22}). \tag{123}$$

This equation expresses the balance between the amounts of heat brought into an element of volume by turbulence and that diffused out by conductivity (cf. Corrsin (1951), where the fluctuations of temperature in homogeneous isotropic turbulence is considered).

6. The equations governing stationary turbulence

In the remaining part of this paper we shall suppose that the field of turbulence is *stationary*. In assuming this, we have in mind the following situation: An external agency maintains a constant mean adverse temperature gradient $-|\beta|$; in order to maintain this gradient energy must be supplied at a constant rate as was the case, for example, in the experiments of Schmidt & Milverton (1935) and Schmidt & Saunders (1938). When an adverse temperature gradient is maintained there will be turbulence and all the field quantities will be subject to fluctuations; but under stationary conditions the statistical properties of the fluctuations described, for example, by the various correlation functions and their defining scalars will be constant with time. In particular, the mean square velocities parallel and perpendicular to the direction $\boldsymbol{\lambda}$ will be constant with time; we can then conclude from equations (120) to (122) that

$$-\frac{g}{\rho_0}\overline{\delta\rho u_\parallel}+2\varpi_{00}=2\nu(10\alpha_{02}+2\alpha_{22}),\quad -2\varpi_{00}=4\nu(10\alpha_{02}+4\alpha_{22}+5\beta_{02}+\beta_{22}) \tag{124}$$

and
$$2\gamma\lambda_{00}=-\frac{g}{\rho_0}\overline{\delta\rho u_\parallel}=4\nu(15\alpha_{02}+5\alpha_{22}+5\beta_{02}+\beta_{22}). \tag{125}$$

Equation (125) states that *under stationary conditions the rate of dissipation of kinetic energy by viscosity is exactly the same as the rate of liberation of potential energy by gravity*. This is an important respect in which the theory of stationary turbulence developed on the premises of this paper differs from the theory developed in connexion with Kolmogoroff's theory of local isotropy. Thus, when translating the equation of von Karman & Howarth into the framework of local isotropy (cf. Batchelor 1947) one does not set $\partial Q/\partial t=0$ (where Q is the scalar defining under conditions of isotropy the tensor $\overline{u_i u_j'}$); one sets it equal to $\frac{1}{3}\epsilon$ where ϵ is the (assumed) constant rate of dissipation of energy by viscosity. The reason why one has to do this is that in the usual manner of formulating the equations of the problem, the source from which turbulence derives its energy is not included or specified in detail; but one supposes that by restricting oneself to 'eddies small compared with the largest present' one can avoid specifying it and introduce the constant rate of energy supply, ϵ, as a parameter of the problem. In our case we do not need to do this, since by assuming that a (constant) mean adverse temperature gradient is being maintained we have in effect specified the source of turbulent energy and included it in the equations.

According to the remarks of the foregoing paragraph we can obtain the equations governing stationary turbulence by setting the time derivatives of all the scalars equal to zero. We thus obtain from equations (97) to (104) the following system of equations:

$$\kappa\Delta_3\Theta=-|\beta|\,[r^2(1-\mu^2)\,D_r+2]\,\Lambda, \tag{126}$$

$$(\kappa+\nu)\,\Delta_5\Lambda=|\beta|\,Q_1-X, \tag{127}$$

$$\nu\Delta_5 Q_1=[r^2(1-\mu^2)\,D_r+4]\,Y, \tag{128}$$

$$\Delta_5 X=\gamma D_r\Theta, \tag{129}$$

$$\Delta_7 Y=\gamma D_r\Lambda, \tag{130}$$

$$D_r\Phi=-(r\mu D_r+D_\mu)\,X,\quad D_\mu\Phi+\gamma\Theta=(r^2D_r+r\mu D_\mu+2)\,X, \tag{131}$$

$$D_r\Pi=-(r\mu D_r+D_\mu)\,Y,\quad D_\mu\Pi+\gamma\Lambda=(r^2D_r+r\mu D_\mu+4)\,Y, \tag{132}$$

$$D_\mu[(r^2D_r+2)\,\Pi+r\mu(D_\mu\Pi+\gamma\Lambda)]+\gamma\Lambda+\nu(\Delta_5 Q_2+2D_{\mu\mu}Q_1)=0, \tag{133}$$

$$(\kappa+\nu)\,\Delta_5 L=0\quad\text{and}\quad\Delta_5 P=-\gamma(r\mu D_r+D_\mu)\,L. \tag{134}$$

It should be emphasized that our present reasons for setting the time derivatives of the various scalars equal to zero are entirely different from those of Rayleigh and Jeffreys for setting $\partial/\partial t = 0$ in their discussion of the stability problem. As has been explained, especially by Jeffreys (1926, 1928), the arguments for their setting $\partial/\partial t = 0$ in the equations of motion and heat conduction are based on the principle of 'exchange of stabilities' according to which, situations in which small perturbations are amplified with time and situations in which they are damped with time form a 'linear sequence' separated by situations in which marginal stability obtains and $\partial/\partial t = 0$. Conversely, it follows from these same arguments that Jeffreys's differential equation (22) is strictly limited in its application to only those situations which are in marginal stability. No such limitation restricts the application of equations (126) to (134) since no considerations of stability are involved in setting the time derivates of the scalars equal to zero.

7. The reduction of the equations of stationary turbulence

In this section we shall derive by successive elimination a single differential equation of the sixth order governing Λ. We shall also obtain some general integrals of the equations (126) to (134).

First, we may observe that according to equations (134) both L and P should vanish identically. This follows from the fact that L satisfies Laplace's equation (in five dimensions); it must therefore vanish if it is bounded everywhere and tends to zero at infinity; P then satisfies Laplace's equation and by the same arguments it must also vanish identically. Thus

$$L \equiv P \equiv 0; \tag{135}$$

therefore, in stationary turbulence (cf. table 1)

$$\overline{\theta u_i'} = \overline{\theta' u_i} \quad \text{and} \quad \overline{\varpi u_i'} = -\overline{\varpi' u_i}, \tag{136}$$

and

$$\Lambda_i = \overline{\theta u_i'} \quad \text{and} \quad \Pi_i = \overline{\varpi u_i'}. \tag{137}$$

Now apply the operator Δ_5 to equation (127) and make use of equations (128) and (129). We obtain

$$\begin{aligned}(\kappa+\nu)\,\Delta_5^2\,\Lambda &= |\beta|\,\Delta_5 Q_1 - \Delta_5 X \\ &= \frac{|\beta|}{\nu}[r^2(1-\mu^2)\,D_r+4]\,Y - \gamma D_r\,\Theta.\end{aligned} \tag{138}$$

Apply Δ_5 once more to this equation. Then

$$(\kappa+\nu)\,\Delta_5^3\,\Lambda = \frac{|\beta|}{\nu}\Delta_5[r^2(1-\mu^2)\,D_r+4]\,Y - \gamma\Delta_5 D_r\,\Theta. \tag{139}$$

The terms on the right-hand side of this equation can be simplified by making use of the identity

$$\Delta_n D_r = D_r\,\Delta_{n-2} \tag{140}$$

which follows from the definitions of these operators. Thus,

$$\gamma\Delta_5 D_r\,\Theta = \gamma D_r\,\Delta_3\,\Theta = -\frac{|\beta|\,\gamma}{\kappa}D_r[r^2(1-\mu^2)\,D_r+2]\,\Lambda, \tag{141}$$

where we have made use of equation (126). Remembering that D_r permutes with any function of $r\mu$ (A.T., equation (9)) and that, as operators,

$$D_r r^2 = r^2 D_r + 2, \tag{142}$$

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we can rewrite equation (141) in the form

$$\gamma \Delta_5 D_r \Theta = -\frac{|\beta|\gamma}{\kappa} D_{rr}[r^2(1-\mu^2)\,\Lambda]. \tag{143}$$

Considering next the first term on the right-hand side of equation (139) we have (cf. equations (140) and (142))

$$\begin{aligned}\Delta_5[r^2(1-\mu^2)\,D_r+4]\,Y &= \Delta_5 D_r[r^2(1-\mu^2)\,Y]+2\Delta_5 Y\\ &= D_r\Delta_3[r^2(1-\mu^2)\,Y]+2\Delta_5 Y.\end{aligned} \tag{144}$$

Writing Δ_3 in the form, $\Delta_3 = D_\mu(r\mu D_r+D_\mu)+D_r(r^2D_r+r\mu D_\mu)$, (145)

and remembering that $r\mu D_r+D_\mu$ permutes with any function of $r^2(1-\mu^2)$ (A.T. equations (18) and (19)) and D_r permutes with any function of $r\mu$, we find on making further use of (142), that

$$\begin{aligned}\Delta_3[r^2(1-\mu^2)\,Y] &= D_\mu[r^2(1-\mu^2)\,(r\mu D_r+D_\mu)\,Y]+D_r[r^2(1-\mu^2)\,(r^2D_r+r\mu D_\mu+2)\,Y]\\ &= r^2(1-\mu^2)\,D_\mu(r\mu D_r+D_\mu)\,Y-2r\mu(r\mu D_r+D_\mu)\,Y\\ &\quad +r^2(1-\mu^2)\,D_r(r^2D_r+r\mu D_\mu+2)\,Y+2(r^2D_r+r\mu D_\mu+2)\,Y\\ &= r^2(1-\mu^2)\,\Delta_5 Y+2r^2(1-\mu^2)\,D_r Y+4Y\\ &= r^2(1-\mu^2)\,\Delta_7 Y+4Y.\end{aligned} \tag{146}$$

Hence

$$\begin{aligned}\Delta_5[r^2(1-\mu^2)\,D_r+4]\,Y &= D_r[r^2(1-\mu^2)\,\Delta_7 Y]+4D_r Y+2\Delta_5 Y\\ &= D_r[r^2(1-\mu^2)\,\Delta_7 Y]+2\Delta_7 Y.\end{aligned} \tag{147}$$

Now replacing $\Delta_7 Y$ by $\gamma D_r \Lambda$ in accordance with equation (130), we have

$$\begin{aligned}\Delta_5[r^2(1-\mu^2)\,D_r+4]\,Y &= \gamma D_r[r^2(1-\mu^2)\,D_r\Lambda]+2\gamma D_r\Lambda\\ &= \gamma D_{rr}[r^2(1-\mu^2)\,\Lambda].\end{aligned} \tag{148}$$

Finally, combining equations (139), (143) and (148) we have

$$(\kappa+\nu)\,\Delta_5^3\Lambda = |\beta|\,\gamma\left(\frac{1}{\kappa}+\frac{1}{\nu}\right)D_{rr}[r^2(1-\mu^2)\,\Lambda], \tag{149}$$

or

$$\Delta_5^3\Lambda = \frac{|\beta|\gamma}{\kappa\nu}D_{rr}[r^2(1-\mu^2)\,\Lambda]. \tag{150}$$

This is the required differential equation for Λ. The similarity of equation (150) with Jeffreys's differential equation (22) (and in particular the occurrence of the five-dimensional operator Δ_5 in place of the Laplacian ∇^2) may be noted.

Some further integrals of equations (126) to (129) may now be derived. Comparing equations (143) and (150) we have

$$\gamma\Delta_5 D_r\Theta = -\nu\Delta_5^3\Lambda, \tag{151}$$

or

$$\Delta_5[\gamma D_r\Theta+\nu\Delta_5^2\Lambda] = 0. \tag{152}$$

If the quantity in brackets is bounded and vanishes at infinity, it must vanish identically. Hence (cf. equation (129))

$$\nu\Delta_5^2\Lambda = -\gamma D_r\Theta = -\Delta_5 X, \tag{153}$$

or

$$\Delta_5[\nu\Delta_5\Lambda+X] = 0; \tag{154}$$

and by the same arguments as before

$$X = -\nu \Delta_5 \Lambda. \tag{155}$$

Substituting this last relation in equation (127) we find that

$$|\beta| \, Q_1 = \kappa \Delta_5 \Lambda; \tag{156}$$

therefore

$$Q_1 = -\frac{\kappa}{|\beta| \, \nu} X. \tag{157}$$

It is evident from the foregoing reductions that once equation (150) for Λ has been solved, the solution for the other scalars can be found successively.

8. The expression of Λ in terms of Q_1

The equation,

$$\Delta_5 \Lambda = \frac{|\beta|}{\kappa} Q_1, \tag{158}$$

derived in the preceding section (equation (156)) is Poisson's equation governing an axially symmetric distribution of matter in a five-dimensional Euclidean space. Since in a five-dimensional space the Newtonian potential $1/r$ is replaced by $1/3r^3$, we can write the general solution of equation (158) in the form

$$\Lambda(\mathbf{r}) = -\frac{|\beta|}{8\pi^2\kappa} \iiiint\!\!\int \frac{Q_1(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \, dx_1' \, dx_2' \, dx_3' \, dx_4' \, dx_5', \tag{159}$$

where $\mathbf{r}$ denotes the radius vector in the five-dimensional space $(x_1, \ldots, x_5)$ and the integration is extended over the entire space. Using polar co-ordinates (for the notation see Chandrasekhar 1950*b*, equation (14)) and remembering that in the case on hand both Λ and Q_1 are functions only of $r = |\mathbf{r}|$ and θ $(=\cos^{-1}\mu)$ we can rewrite equation (159) in the form

$$\Lambda(r,\theta) = -\frac{|\beta|}{8\pi^2\kappa} \int_0^\infty \int_0^\pi \int_0^\pi \int_0^\pi \int_0^{2\pi} \frac{Q_1(r',\theta')}{|\mathbf{r}-\mathbf{r}'|^3} r'^4 \sin^3\theta' \, \sin^2\phi_1' \, \sin\phi_2' \, dr' \, d\theta' \, d\phi_1' \, d\phi_2' \, d\phi_3'. \tag{160}$$

The integrations over ϕ_2' and ϕ_3' are readily performed and we are left with

$$\Lambda(r,\theta) = -\frac{|\beta|}{2\pi\kappa} \int_0^\infty \int_0^\pi \int_0^\pi \frac{Q_1(r',\theta')}{|\mathbf{r}-\mathbf{r}'|^3} r'^4 \sin^3\theta' \, \sin^2\phi_1' \, dr' \, d\theta' \, d\phi_1'. \tag{161}$$

We now expand $|\mathbf{r}-\mathbf{r}'|^{-3}$ in terms of the Gegenbauer polynomials $C_m^{\frac{3}{2}}(\cos\Theta)$, where Θ denotes the angle between the directions specified by $(\theta, 0)$ and (θ', ϕ_1'):

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{1}{\tilde{r}^3} \sum_{m=0}^{\infty} C_m^{\frac{3}{2}}(\cos\Theta) \, x^m, \tag{162}$$

where $\tilde{r}$ is the larger of r and r' and $x = r/r'$, or r'/r, whichever is less than one. Inserting (162) in (161) and using the relation given in Watson (1944, p. 369) we obtain

$$\Lambda(r,\mu) = -\frac{|\beta|}{2\pi\kappa} \int_0^\infty \int_{-1}^{+1} Q_1(r',\mu') \, (1-\mu'^2) \frac{r'^4}{\tilde{r}^3} \sum_{m=0}^{\infty} \frac{2^2 m! [\Gamma(\frac{3}{2})]^2}{\Gamma(m+3)} C_m^{\frac{3}{2}}(\mu) \, C_m^{\frac{3}{2}}(\mu') \, x^m \, dr' \, d\mu'. \tag{163}$$

Without loss of any essential generality, we may suppose that $Q_1(r,\mu)$ can be expanded as a series in Gegenbauer polynomials $C_m(\mu)$; however, since Q_1 is required to be an even function of r and μ the expansion will contain only the even order polynomials $C^{\frac{3}{2}}_{2n}(\mu)$. Let, therefore,

$$Q_1(r,\mu) = \sum_{n=0}^{\infty} Q_1^{(2n)}(r)\, C^{\frac{3}{2}}_{2n}(\mu). \tag{164}$$

Introducing this expansion in (163) and using the orthogonality property of the Gegenbauer polynomials (cf. Sommerfeld 1949, p. 232) we find:

$$\Lambda(r,\mu) = -\frac{|\beta|}{2\kappa} \sum_{n=0}^{\infty} \frac{C^{\frac{3}{2}}_{2n}(\mu)}{2n+\frac{3}{2}} \int_0^{\infty} Q_1^{(2n)}(r')\, \frac{r'^4}{\bar{r}^3}\, x^{2n}\, \mathrm{d}r', \tag{165}$$

or, somewhat more explicitly,

$$\Lambda(r,\mu) = -\frac{|\beta|}{2\kappa} \sum_{n=0}^{\infty} \frac{C^{\frac{3}{2}}_{2n}(\mu)}{2n+\frac{3}{2}} \left\{ r^{2n} \int_r^{\infty} \frac{Q_1^{(2n)}(r')}{r'^{2n-1}}\, \mathrm{d}r' + \frac{1}{r^{2n+3}} \int_0^r Q_1^{(2n)}(r')\, r'^{2n+4}\, \mathrm{d}r' \right\}. \tag{166}$$

Equation (166) expresses Λ in terms of Q_1.

Setting $r = 0$ in equation (166), we obtain

$$\Lambda(0) = -\frac{|\beta|}{3\kappa} \int_0^{\infty} Q_1(r)\, r\, \mathrm{d}r = -\frac{|\beta|}{6\kappa} \int_0^{\infty}\int_{-1}^{+1} Q_1(r,\mu)\, r\, \mathrm{d}r\, \mathrm{d}\mu. \tag{167}$$

Since the constant rate of dissipation of energy per unit mass is given by $2\gamma\Lambda(0)$ (cf. equation (125)), we can write

$$\epsilon = -\frac{|\beta|\,\gamma}{3\kappa} \int_0^{\infty}\int_{-1}^{+1} Q_1(r,\mu)\, r\, \mathrm{d}r\, \mathrm{d}\mu. \tag{168}$$

According to equation (166) the angle independent term, $\Lambda_0(r)$ in the expansion of $\Lambda(r,\mu)$ in Gegenbauer polynomials is given by

$$\Lambda_0(r) = -\frac{|\beta|}{3\kappa} \left\{ \int_r^{\infty} Q_1^{(0)}(r')\, r'\, \mathrm{d}r' + \frac{1}{r^3} \int_0^r Q_1^{(0)}(r')\, r'^4\, \mathrm{d}r' \right\}. \tag{169}$$

From this equation it follows that if $\Lambda r^3 \to 0$ as $r \to \infty$, then

$$\int_0^{\infty} Q_1^{(0)}(r)\, r^4\, \mathrm{d}r = 0. \tag{170}$$

This is a kind of Loitsiansky invariant for the problem on hand. This may mean that under the conditions in which we are discussing the problem (and in particular in consequence of the neglect of the inertial term in the equation of motion) the turbulent energy is not stored, principally, among the very large eddies as is generally assumed to be the case.

9. The fundamental solutions of equations (126) to (133)

As we have seen in § 7 the solution of the system of equations (126) to (133) can be reduced to that of the single equation (equation (150))

$$\Delta_5^3 \Lambda = \frac{|\beta|\,\gamma}{\kappa\nu} D_{rr}[r^2(1-\mu^2)\Lambda]. \tag{171}$$

From general considerations it would appear that in the direction of $\boldsymbol{\lambda}$ ($\mu = 1$) we should be able to express Λ as a Fourier cosine integral (cosine integral since Λ is an even function of r and μ) in the form

$$\Lambda(r=z;\, \mu=1) = \int_0^{\infty} L(a) \cos az\, \mathrm{d}a. \tag{172}$$

We shall accordingly seek a solution of equation (171) which is separable in the variables

$$z = r\mu \quad \text{and} \quad y = r(1-\mu^2)^{\frac{1}{2}}, \tag{173}$$

and is, indeed, of the form

$$\Lambda(r,\mu) = R(y)\cos az, \tag{174}$$

where $R(y)$ is a function of y only. Once a solution of this form has been obtained, we can, by superposing solutions with different a's, obtain the general solution.

In reducing the differential equation (171) when Λ has the form (174) and more generally in the further treatment of the other equations of the problem, we shall find it convenient to have for reference table 2 which lists the effects of the various differential operators of the theory on functions which depend only on y or z. And using the results of this table we may verify that

$$\Delta_n R(y)\, Z(z) = [D_\mu(r\mu D_r + D_\mu) + (r^2 D_r + r\mu D_\mu + n - 1)\, D_r]\, R(y)\, Z(z)$$
$$= Z''R + Z\left(R'' + \frac{n-2}{y}R'\right). \tag{175}$$

Table 2. The effects of the various differential operators on functions depending only on y or z

$D_r Z(z) = 0$	$D_r R(y) = \dfrac{R'}{y}$
$D_\mu Z(z) = Z'$	$D_\mu R(y) = -\dfrac{z}{y} R'$
$(r\mu D_r + D_\mu)\, Z(z) = Z'$	$(r\mu D_r + D_\mu)\, R(y) = 0$
$(r^2 D_r + r\mu D_\mu)\, Z(z) = zZ'$	$(r^2 D_r + r\mu D_\mu)\, R(y) = yR'$
$\Delta_n Z(z) = Z''$	$\Delta_n R(y) = R'' + \dfrac{n-2}{y} R'$

(Primes denote differentiation with respect to the argument of the function: y in the case of R and z in the case of Z.)

Similarly, we also find

$$D_{rr}[r^2(1-\mu^2)\, R(y)\, Z(z)] = ZD_r[D_r(y^2R)] = ZD_r(R'y + 2R) = Z\left(R'' + \frac{3}{y}R'\right). \tag{176}$$

Returning to equation (171) and using the results expressed by equations (175) and (176), we find that for Λ of the form (174), it reduces to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{3}{y}\frac{\mathrm{d}}{\mathrm{d}y} - a^2\right)^3 R(y) = \frac{|\beta|\,\gamma}{\kappa\nu}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{3}{y}\frac{\mathrm{d}}{\mathrm{d}y}\right) R(y). \tag{177}$$

Now it may be readily verified that

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{3}{y}\frac{\mathrm{d}}{\mathrm{d}y}\right)\frac{J_1(\alpha y)}{\alpha y} = -\alpha^2 \frac{J_1(\alpha y)}{\alpha y}, \tag{178}$$

where α is an arbitrary constant and J_1 is a Bessel function (of the first kind) of order 1. [It may be noticed that we have incidentally proved that (cf. equations (175) and (178))

$$\Delta_5 \frac{J_1(\alpha y)}{\alpha y}\cos az = -(a^2 + \alpha^2)\frac{J_1(\alpha y)}{\alpha y}\cos az. \tag{179}$$

More generally, it is true that

$$\Delta_{2n+1} \frac{J_{n-1}(\alpha y)}{(\alpha y)^{n-1}} e^{iaz} = -(a^2+\alpha^2) \frac{J_{n-1}(\alpha y)}{(\alpha y)^{n-1}} e^{iaz} \tag{180)]}$$

From equations (177) and (178) it follows that

$$R(y) = \frac{J_1(\alpha y)}{\alpha y}$$

is a solution of equation (177) provided (cf. equation (27))

$$\frac{|\beta|\gamma}{\kappa\nu} = \frac{(a^2+\alpha^2)^3}{\alpha^2}. \tag{182}$$

For a given a^2 the roots of this cubic equation for α^2 can be expressed parametrically in the form (cf. Low 1929 and Hales 1937)

$$\frac{|\beta|\gamma}{\kappa\nu} = \frac{(c^2+3)^3}{4(c^2-1)^2} a^4 \tag{183}$$

and

$$\frac{\alpha_1^2}{a^2} = \frac{4}{c^2-1}, \quad \frac{\alpha_2^2}{a^2} = \frac{(c-1)^2}{2(c+1)}, \quad \frac{\alpha_3^2}{a^2} = -\frac{(c+1)^2}{2(c-1)}, \tag{184}$$

where c is a pure number. On examination it is found that the ranges $c = \infty$ to 3, $c = 3$ to 1, $c = 1$ to 0, $c = 0$ to -1, $c = -1$ to -3 and $c = -3$ to $-\infty$ give the same set of roots (though they are permuted among themselves). Without loss of generality we may, therefore, restrict c to the range $3 \leqslant c < \infty$. With c thus restricted, α_1^2 and α_2^2 are positive while α_3^2 is negative. The solutions derived from this last negative root of α^2 cannot be used since the (modified) Bessel functions I_1 and K_1 have singularities: the former at infinity and the latter at the origin. For the same reason the solutions Y_1 for the positive roots α_1^2 and α_2^2 cannot also be used. The general solution of Λ of the form (174) is, therefore, given by

$$\Lambda(r,\mu) = \left\{\sum_{i=1}^{2} K_i \frac{J_1(\eta_i)}{\eta_i}\right\} \cos az, \tag{185}$$

where, for brevity, we have written

$$\eta_i = \alpha_i y \quad (i = 1, 2) \tag{186}$$

and K_1 and K_2 are two arbitrary constants.

In table 3 we have listed the values of α_1^2/a^2, α_2^2/a^2 and $|\beta|\gamma/\kappa\nu a^4$ for various values of c.

TABLE 3. THE ROOTS OF THE CUBIC EQUATION (182) FOR α^2

c	α_1^2/a^2	α_2^2/a^2	$\|\beta\|\gamma/\kappa\nu a^4$	c	α_1^2/a^2	α_2^2/a^2	$\|\beta\|\gamma/\kappa\nu a^4$
3·0	0·5000	0·5000	6·750	6·0	0·1143	1·786	12·11
3·2	0·4329	0·5762	6·796	6·5	0·09697	2·017	13·61
3·4	0·3788	0·6545	6·920	7·0	0·08333	2·250	15·26
3·6	0·3344	0·7348	7·105	7·5	0·07240	2·485	17·03
3·8	0·2976	0·8167	7·342	8·0	0·06349	2·722	18·94
4·0	0·2667	0·9000	7·621	8·5	0·05614	2·960	20·98
4·2	0·2404	0·9846	7·939	9·0	0·05000	3·200	23·15
4·4	0·2179	1·070	8·291	9·5	0·04482	3·440	25·45
4·6	0·1984	1·157	8·675	10·0	0·04040	3·682	27·87
4·8	0·1815	1·245	9·087	15·0	0·01785	6·125	59·05
5·0	0·1667	1·333	9·528	20·0	0·01002	8·595	102·8
5·5	0·1368	1·558	10·74	25·0	0·006410	11·08	159·0

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With Λ given by equation (185), the corresponding solutions of the other scalars can be found successively. Thus, from equations (155) and (157), it follows at once that (cf. equation (179))

$$X = -\nu\Delta_5\Lambda = \nu\left\{\sum_{i=1}^{2} K_i(a^2+\alpha_i^2)\frac{J_1(\eta_i)}{\eta_i}\right\}\cos az \tag{187}$$

and

$$Q_1 = -\frac{\kappa}{|\beta|}\left\{\sum_{i=1}^{2} K_i(a^2+\alpha_i^2)\frac{J_1(\eta_i)}{\eta_i}\right\}\cos az. \tag{188}$$

Next, we determine Θ from equation (153). Thus

$$D_r\Theta = -\frac{\nu}{\gamma}\Delta_5^2\Lambda = -\frac{\nu}{\gamma}\left\{\sum_{i=1}^{2} K_i(a^2+\alpha_i^2)^2\frac{J_1(\eta_i)}{\eta_i}\right\}\cos az. \tag{189}$$

On the other hand (cf. table 2)

$$D_r J_0(\alpha y)\cos az = \cos az\frac{1}{y}\frac{d}{dy}J_0(\alpha y) = -\alpha^2\frac{J_1(\alpha y)}{\alpha y}\cos az. \tag{190}$$

Hence the solution for Θ is

$$\Theta = \frac{\nu}{\gamma}\left\{\sum_{i=1}^{2}\frac{K_i}{\alpha_i^2}(a^2+\alpha_i^2)^2 J_0(\eta_i)\right\}\cos az, \tag{191}$$

or using equation (182) we can rewrite it in the form

$$\Theta = \frac{|\beta|}{\kappa}\left\{\sum_{i=1}^{2}\frac{K_i}{a^2+\alpha_i^2}J_0(\eta_i)\right\}\cos az. \tag{192}$$

To determine Y we use equation (130). Thus

$$\Delta_7 Y = \gamma D_r\Lambda = \gamma\left\{\sum_{i=1}^{2} K_i\alpha_i^2\frac{1}{\eta_i}\frac{d}{d\eta_i}\frac{J_1(\eta_i)}{\eta_i}\right\}\cos az$$

$$= -\gamma\left\{\sum_{i=1}^{2} K_i\alpha_i^2\frac{J_2(\eta_i)}{\eta_i^2}\right\}\cos az. \tag{193}$$

On the other hand (cf. equation (180))

$$\Delta_7\frac{J_2(\alpha y)}{(\alpha y)^2}\cos az = -(a^2+\alpha^2)\frac{J_2(\alpha y)}{(\alpha y)^2}\cos az. \tag{194}$$

Hence the solution for Y is

$$Y = \gamma\left\{\sum_{i=1}^{2}\frac{K_i\alpha_i^2}{a^2+\alpha_i^2}\frac{J_2(\eta_i)}{\eta_i^2}\right\}\cos az. \tag{195}$$

Now combining equations (47) and (191) we obtain (cf. table 2)

$$\Delta_3\Phi = a\nu\left\{\sum_{i=1}^{2}\frac{K_i}{\alpha_i^2}(a^2+\alpha_i^2)^2 J_0(\eta_i)\right\}\sin az. \tag{196}$$

But (cf. equation (180))

$$\Delta_3 J_0(\alpha y)\sin az = -(a^2+\alpha^2)J_0(\alpha y)\sin az. \tag{197}$$

Hence

$$\Phi = -a\nu\left\{\sum_{i=1}^{2}\frac{K_i}{\alpha_i^2}(a^2+\alpha_i^2)J_0(\eta_i)\right\}\sin az. \tag{198}$$

Similarly, from equations (91) and (185) we find that

$$\Pi = -a\gamma\left\{\sum_{i=1}^{2}\frac{K_i}{a^2+\alpha_i^2}\frac{J_1(\eta_i)}{\eta_i}\right\}\sin az. \tag{199}$$

It remains to solve equation (133) for Q_2. Using equation (132) we can rewrite equation (133) in the form

$$-\nu\Delta_5 Q_2 = D_\mu(-y^2 D_\mu Y + 4zY + 2\Pi) + \gamma\Lambda + 2\nu D_{\mu\mu} Q_1. \tag{200}$$

Substituting for Q_1, Y and Π according to equations (188), (195) and (199) in the foregoing equations we find after some lengthy reductions that

$$\begin{aligned}-\nu\Delta_5 Q_2 = \gamma\sum_{i=1}^{2} K_i\Bigg\{&\left[\frac{2a^2\alpha_i^2}{(a^2+\alpha_i^2)^2}+1\right]\frac{J_1(\eta_i)}{\eta_i}\cos az\\ &+\frac{1}{a^2+\alpha_i^2}\left[-a^2 J_0(\eta_i)\cos az - 2a\alpha_i^2\frac{J_1(\eta_i)}{\eta_i}z\sin az + \alpha_i^2\frac{J_1(\eta_i)}{\eta_i}\cos az\right]\\ &+\frac{\alpha_i^4}{a^2+\alpha_i^2}\left[\left(\frac{J_2(\eta_i)}{\eta_i^2}-\frac{2\alpha_i^2}{a^2+\alpha_i^2}\frac{J_3(\eta_i)}{\eta_i^3}\right)z^2\cos az + \frac{2}{a^2+\alpha_i^2}\frac{J_2(\eta_i)}{\eta_i^2}(2az\sin az - \cos az)\right]\Bigg\}. \end{aligned}\tag{201}$$

The right-hand side of equation (201) can be expressed as the result of operation of Δ_5 on a known function. For this purpose we need the following relations which can be established in a fairly straightforward manner:

$$\left.\begin{aligned}\Delta_5\frac{J_2(\alpha y)}{(\alpha y)^2}z^2\cos az &= \left[-(a^2+\alpha^2)\frac{J_2(\alpha y)}{(\alpha y)^2}+2\alpha^2\frac{J_3(\alpha y)}{(\alpha y)^3}\right]z^2\cos az\\ &\qquad -2\frac{J_2(\alpha y)}{(\alpha y)^2}(2az\sin az-\cos az),\\ \Delta_5 J_0(\alpha y)\cos az &= -(a^2+\alpha^2)J_0(\alpha y)\cos az - 2\alpha^2\frac{J_1(\alpha y)}{\alpha y}\cos az,\\ \Delta_5\frac{J_1(\alpha y)}{\alpha y}z\sin az &= -(a^2+\alpha^2)\frac{J_1(\alpha y)}{\alpha y}z\sin az + 2a\frac{J_1(\alpha y)}{\alpha y}\cos az.\end{aligned}\right\}\tag{202}$$

Using these relations and also (179), we find that we can reduce equation (201) to the form

$$\begin{aligned}-\nu\Delta_5 Q_2 = \gamma\sum_{i=1}^{2}\frac{K_i}{(a^2+\alpha_i^2)^2}\Delta_5\Bigg\{&-\alpha_i^4\frac{J_2(\eta_i)}{\eta_i^2}z^2\cos az + 2a\alpha_i^2\frac{J_1(\eta_i)}{\eta_i}z\sin az\\ &-(a^2+2\alpha_i^2)\frac{J_1(\eta_i)}{\eta_i}\cos az + a^2 J_0(\eta_i)\cos az\Bigg\}.\end{aligned}\tag{203}$$

Hence the required solution for Q_2 is

$$\begin{aligned}Q_2 = -\frac{\kappa}{|\beta|}\sum_{i=1}^{2}\frac{K_i}{\alpha_i^2}(a^2+\alpha_i^2)\Bigg\{&a^2 J_0(\eta_i)\cos az - (a^2+2\alpha_i^2)\frac{J_1(\eta_i)}{\eta_i}\cos az\\ &+2a\alpha_i^2\frac{J_1(\eta_i)}{\eta_i}z\sin az - \alpha_i^4\frac{J_2(\eta_i)}{\eta_i^2}z^2\cos az\Bigg\},\end{aligned}\tag{204}$$

where we have further replaced γ/ν by $(\kappa/|\beta|)(a^2+\alpha_i^2)^3/\alpha_i^2$ in accordance with equation (182).

This completes the solution of equations (126) to (133). As for equations (134), we have already seen that both L and P are identically zero (equation (135)).

10. The correlation tensor Q_{ij}; the mean energy of turbulence, and the dissipation by viscosity

Explicit expressions for the coefficients of the correlation tensor (cf. A.T., equation (48)),

$$Q_{ij} = A\xi_i\xi_j + B\delta_{ij} + C\lambda_i\lambda_j + D(\xi_i\lambda_j + \lambda_i\xi_j), \tag{205}$$

can be found by inserting in A.T., equations (50), the solutions (188) and (204) for Q_1 and Q_2. Thus, we find after some lengthy reductions that

$$\left.\begin{aligned} A &= -\frac{\kappa}{|\beta|}a^2\left\{\sum_{i=1}^{2}K_i(a^2+\alpha_i^2)\frac{J_2(\eta_i)}{\eta_i^2}\right\}\cos az,\\ B &= +\frac{\kappa}{|\beta|}a^2\left\{\sum_{i=1}^{2}\frac{K_i}{\alpha_i^2}(a^2+\alpha_i^2)\frac{J_1(\eta_i)}{\eta_i}\right\}\cos az,\\ C &= -\frac{\kappa}{|\beta|}\sum_{i=1}^{2}K_i(a^2+\alpha_i^2)\left\{2a\frac{J_1(\eta_i)}{\eta_i}z\sin az\right.\\ &\qquad\left.+\left[a^2r^2\frac{J_2(\eta_i)}{\eta_i^2}+\frac{a^2}{\alpha_i^2}J_0(\eta_i)-\left(2+\frac{a^2}{\alpha_i^2}\right)\frac{J_1(\eta_i)}{\eta_i}+J_2(\eta_i)\right]\cos az\right\},\\ D &= +\frac{\kappa}{|\beta|}\sum_{i=1}^{2}K_i(a^2+\alpha_i^2)\left\{a^2\frac{J_2(\eta_i)}{\eta_i^2}z\cos az+a\frac{J_1(\eta_i)}{\eta_i}\sin az\right\}. \end{aligned}\right\} \tag{206}$$

A quantity of some interest is the correlation $\overline{u_\parallel(0)\,u'_\parallel(z)}$ of the velocities parallel to $\boldsymbol{\lambda}$ at two points vertically above one another and at a distance z apart. This correlation can be found by setting $i=j$ (no summation), $\lambda_i = 1$, $\xi_i = z$ and $\mu = 1$ in equation (205). Thus

$$\overline{u_\parallel(0)\,u'_\parallel(z)} = [Az^2+B+C+2zD]_{\mu=1,\,r=z}. \tag{207}$$

Evaluating the quantity on the right-hand side in accordance with equations (206) we find the simple result:

$$\overline{u_\parallel(0)\,u'_\parallel(z)} = \frac{\kappa}{|\beta|}\left\{\sum_{i=1}^{2}K_i(a^2+\alpha_i^2)\right\}\cos az. \tag{208}$$

It is thus seen that in seeking a solution for Λ of the form (174) and considering a superposition of the solutions with different a's we are, in effect, expressing the correlation $\overline{u_\parallel(0)\,u'_\parallel(z)}$ as a Fourier cosine integral.

Setting $z=0$ in (208), we obtain

$$\overline{u_\parallel^2} = \frac{\kappa}{|\beta|}\sum_{i=1}^{2}K_i(a^2+\alpha_i^2). \tag{209}$$

The corresponding expression for the mean square velocity in the perpendicular direction is clearly given by the value of the coefficient B at the origin; thus

$$\overline{u_\perp^2} = \frac{1}{2}\frac{\kappa}{|\beta|}\sum_{i=1}^{2}K_i(a^2+\alpha_i^2)\frac{a^2}{\alpha_i^2}. \tag{210}$$

Hence (cf. equation (182))

$$\overline{u^2} = \frac{\kappa}{|\beta|}\sum_{i=1}^{2}K_i\frac{(a^2+\alpha_i^2)^2}{\alpha_i^2} = \frac{\gamma}{\nu}\sum_{i=1}^{2}\frac{K_i}{a^2+\alpha_i^2}. \tag{211}$$

Returning to the solutions (188) and (204) for Q_1 and Q_2, we find that for $r\to 0$, they have the series expansions,

$$\left.\begin{aligned} Q_1 &= -\frac{\kappa}{|\beta|}\sum_{i=1}^{2}K_i(a^2+\alpha_i^2)\left[\tfrac{1}{2}-r^2\left\{\frac{\alpha_i^2}{16}+\tfrac{1}{16}(4a^2-\alpha_i^2)\,\mu^2\right\}+\dots\right],\\ Q_2 &= -\frac{\kappa}{|\beta|}\sum_{i=1}^{2}K_i(a^2+\alpha_i^2)\left[\left(\frac{1}{2}\frac{a^2}{\alpha_i^2}-1\right)+r^2\left\{\left(\tfrac{1}{8}\alpha_i^2-\tfrac{3}{16}a^2\right)+\left(\tfrac{27}{16}a^2-\tfrac{1}{4}\alpha_i^2-\frac{1}{4}\frac{a^4}{\alpha_i^2}\right)\mu^2\right\}+\dots\right]. \end{aligned}\right\} \tag{212}$$

From equations (212) we can directly read off the coefficients α_{00}, β_{00}, etc., in the expansions for Q_1 and Q_2 assumed in equations (105); and we can verify that the values of $\overline{u_{\parallel}^2}$ and $\overline{u_{\perp}^2}$ given by equations (106) are in agreement with equations (209) and (210).

The constant rates of dissipation, $\epsilon_{\parallel}$ and $\epsilon_{\perp}$, of the kinetic energies parallel and perpendicular, respectively, to the direction $\boldsymbol{\lambda}$ can now be found in accordance with equations (124). Thus,

$$\epsilon_{\parallel} = 2\nu(10\alpha_{02}+2\alpha_{22}) = \gamma \sum_{i=1}^{2} K_i \frac{\alpha_i^2}{a^2+\alpha_i^2}, \tag{213}$$

$$\epsilon_{\perp} = 4\nu(10\alpha_{02}+4\alpha_{22}+5\beta_{02}+\beta_{22}) = \gamma \sum_{i=1}^{2} K_i \frac{a^2}{a^2+\alpha_i^2}, \tag{214}$$

and

$$\epsilon = \epsilon_{\parallel}+\epsilon_{\perp} = \gamma(K_1+K_2). \tag{215}$$

This last equation for ϵ is in agreement with equation (125) since according to the solution (185) for Λ,

$$2\gamma\Lambda(0) = \gamma(K_1+K_2) = \epsilon. \tag{216}$$

The rate of transfer of kinetic energy from the direction parallel to $\boldsymbol{\lambda}$ to the direction perpendicular to $\boldsymbol{\lambda}$ is given by $-2\varpi_{00}$ (cf. equation (124)) where ϖ_{00} is the coefficient of $r\mu$ in the expansion of Π at the origin; and this according to the solution (199) for Π is

$$-2\varpi_{00} = \gamma a^2 \sum_{i=1}^{2} \frac{K_i}{a^2+\alpha_i^2}. \tag{217}$$

We observe that, in agreement with equation (124), this is equal to $\epsilon_{\perp}$. Also comparing equations (211) and (214) we have

$$\epsilon_{\perp} = -2\varpi_{00} = \nu a^2 \overline{u^2}. \tag{218}$$

In other words, the rate of dissipation of kinetic energy perpendicular to $\boldsymbol{\lambda}$ is proportional to the mean energy of turbulence (in a particular wave-length) and to the square of the wave number of the 'eddy' in the $\boldsymbol{\lambda}$-direction. In homogeneous isotropic turbulence a relation similar to (218) is valid for ϵ; in the case on hand it is true only of $\epsilon_{\perp}$.

Another relation of some interest which follows from the solution for the scalars given in § 9 is the proportionality of the mean energy of turbulence and the mean square fluctuations of temperature: for, from equations (192) and (211) we obtain

$$\overline{\theta^2} = \Theta(0) = \frac{|\beta|}{\kappa} \sum_{i=1}^{2} \frac{K_i}{a^2+\alpha_i^2} = \frac{\nu|\beta|}{\kappa\gamma}\overline{u^2}. \tag{219}$$

An alternative form of this relation is (cf. equation (11))

$$g^2\frac{\overline{\delta^2\rho}}{\rho_0^2} = g^2\alpha^2\overline{\theta^2} = \gamma^2\overline{\theta^2} = \nu^2\frac{|\beta|\gamma}{\kappa\nu}\overline{u^2}. \tag{220}$$

Since this last relation is valid independently of a it follows that it is true quite generally, i.e. for any superposition of the solutions.

11. The wave-length of the smallest eddy present: the initiation of turbulence by thermal instability

As we have already remarked, the general solution of the system of equations (126) to (133) can be obtained by superposing the solutions given in § 9 for different values of a. This procedure corresponds to a generalized Fourier analysis of the various correlation functions. Thus from the relation (208) it follows that the cosine transform of $\overline{u_{\parallel}(0)\,u'_{\parallel}(z)}$ is related simply to the manner in which the solutions given in § 9 must be superposed to obtain the general solution. Now an important consequence of the analysis of the preceding sections is that in such a superposition we cannot include wave-lengths less than a certain minimum value. For, writing equation (182) in the form

$$\frac{|\beta|\,\gamma}{\kappa\nu}=\frac{(\alpha^2/a^2+1)^3}{\alpha^2/a^2}a^4, \tag{221}$$

we conclude that

$$\frac{|\beta|\,\gamma}{\kappa\nu}\geqslant\tfrac{27}{4}a^4, \tag{222}$$

since, for positive x, $(x+1)^3/x$ has a minimum value, $\frac{27}{4}$, which it attains for $x=\frac{1}{2}$. The minimum wave-length which can occur in the superposition of the correlation functions is, therefore, given by

$$\lambda_m^4=108\pi^4\frac{\kappa\nu}{|\beta|\,\gamma}. \tag{223}$$

The origin of this minimum wave-length in the theory becomes clearer when we observe that according to equations (211) and (215) the necessary and sufficient conditions for the dissipation by viscosity and the mean energy of turbulence to vanish are (i) $K_1=-K_2$ and (ii) $\alpha_1^2=\alpha_2^2$. The latter condition implies that (cf. equation (184) and table 3)

$$\alpha_1^2=\alpha_2^2=\tfrac{1}{2}a^2 \quad\text{and}\quad \frac{|\beta|\,\gamma}{\kappa\nu}=\tfrac{27}{4}a^4. \tag{224}$$

We thus recover equation (222) *as the condition for* ϵ *and* $\overline{u^2}$ *to vanish simultaneously.*

It will be noticed that equation (223) is identical with Rayleigh's criterion (29).* The reason for this exact agreement is that by making a Fourier analysis of the velocity field in the case of marginal stability, Rayleigh effectively determined the minimum wave-length for the fluctuations in the z-direction which is possible when a given mean adverse temperature gradient is maintained; and since we are also making a Fourier analysis of the correlation $\overline{u_{\parallel}(0)\,u'_{\parallel}(z)}$ it is clear that periodicities incompatible with Rayleigh's criterion (29) cannot occur in the present analysis. An alternative way of describing the situation is as follows: Consider an extensive medium (stratified in planes perpendicular to the vertical) of depth H in which a constant mean adverse temperature gradient $-|\beta|$ is maintained. By Rayleigh's criterion (2) (or correctly by the criterion derived from Jeffreys's differential equation (22) by satisfying the proper boundary conditions at the top and the bottom surfaces of the fluid) there will be convection in the medium. Consider a portion of the fluid far from the bounding surfaces so that the assumption of homogeneity may be approximately realized. Equation (223) would then apply and has the meaning that in a Fourier analysis of correlations such as $\overline{u_{\parallel}(0)\,u'_{\parallel}(z)}$ wave-lengths less than $\lambda_{\min}$ cannot occur. Also, for $\lambda=\lambda_{\min}$ both ϵ and $\overline{u^2}$ vanish. It is evident now that $\lambda_{\min}$ must agree with Rayleigh's criterion (29) (not (2)). On the other hand, the agreement of (223) with Rayleigh's criterion (29) should not be allowed to obscure the fact that the physical situations contemplated on the two theories are entirely different. As we have already remarked the Rayleigh-

* Notice, however, that we are not comparing it with the criterion (2) given by Rayleigh for the stability of a layer of liquid of height H.

Jeffreys theory applies only to the case of marginal stability; whereas, we are considering the state of affairs when the criterion for stability given by that theory has been far surpassed.

We can now describe in general terms as to what will happen when, maintaining a constant mean adverse temperature gradient, we gradually increase the depth, H, of a layer of fluid; or equivalently increase the numerical value of the temperature gradient while keeping the depth the same; in both cases the

$$\text{Rayleigh number} = \frac{|\beta|\gamma}{\kappa\nu} H^4 \tag{225}$$

is continually increased.

First, we shall of course have stability and the transport of heat through the medium will be by conduction only. Then when marginal stability is reached with the passing of the appropriate Jeffreys's criterion we shall have convection with a cellular pattern. And when the Rayleigh number is still further increased, more and more of the modes of motion that can occur in an infinite medium will become possible and be excited. In the early stages, i.e. soon after the critical Rayleigh's number of marginal stability has been passed, the assumption of homogeneity will be far from being even approximately fulfilled: indeed $H = \lambda_{\min}$ already corresponds to a Rayleigh number $= 1{\cdot}05 \times 10^4$. However, when the Rayleigh number becomes sufficiently large, homogeneous axisymmetric turbulence of the kind postulated will be approximated in a large part of the fluid. And we can then interpret the criterion (222) by saying that under these conditions there will be a smallest size for the eddies present corresponding to the minimum wave-length $\lambda_{\min}$.

The foregoing description of the manner in which the statistical character of the turbulence will become gradually established is in general agreement with the experiments of Schmidt & Saunders (1938). It also explains why the transition between the 'cellular' and the 'turbulent' patterns of convection cannot be a sharp one and why it is only for Rayleigh numbers larger than the critical one for marginal stability by factors exceeding twenty that the random character of turbulence manifests itself.

12. The two modes of turbulence caused by thermal instability

While the description given in § 11 of the initiation of turbulence by thermal instability is in agreement with general ideas on the subject, the mathematical theory which has been developed in the preceding sections is limited in its applications by the fact that the inertial term in the equation of motion has been neglected. An essential element in all turbulence phenomena, namely, the transfer of energy from one Fourier component of the velocity fluctuation to another has been ignored. The linear character of the equations resulting from this neglect of the inertial term implies that each Fourier component evolves independently of the others. Consequently, the energy appropriate to each wave number a must remain indeterminate; and the distribution of energy with a must also remain beyond the scope of the theory. On these grounds, we might have expected that a fundamental set of solutions of equations (126) to (133) would contain a single arbitrary constant. Instead, the solutions given in § 9 actually contain two arbitrary constants corresponding to the two positive roots α_1^2 and α_2^2 of the cubic equation (182) for α^2. The occurrence of these two constants in the solutions implies that there are two principal modes of fluctuations for $\lambda > \lambda_{\min}$

(for $\lambda = \lambda_{min}$ the two modes coincide). For these two modes of turbulence (cf. equations (209) and (210))

$$\left(\frac{\overline{u_{\parallel}^2}}{\overline{u_{\perp}^2}}\right)_{\text{I}} = 2\frac{\alpha_1^2}{a^2} \leqslant 1 \quad \text{and} \quad \left(\frac{\overline{u_{\parallel}^2}}{\overline{u_{\perp}^2}}\right)_{\text{II}} = 2\frac{\alpha_2^2}{a^2} \geqslant 1. \tag{226}$$

For sufficiently long wave-lengths, the kinetic energy of turbulence in the two modes will be confined almost exclusively to the perpendicular or the parallel components of the velocity. It is remarkable that the theory should disclose the existence of two such essentially dissimilar modes of turbulence without giving any indications as to the circumstances under which the one or the other will prevail.

13. Concluding remarks

In concluding this paper we may refer to the analogy (to which Low (1925) first drew attention) between the conditions in a layer of liquid heated below and a liquid between two coaxial cylinders rotating at different rates. The criterion for stability in the latter problem was established both theoretically and experimentally by Taylor (1923). But it was Jeffreys (1928) who provided the mathematical basis of Low's analogy by showing that the equations which govern the situations of marginal stability in the two problems are very similar. On the basis of this analogy we may expect that the problem of the initiation of turbulence in a differentially rotating medium can be treated by the methods of this paper. The author hopes to return to this problem on a later occasion.

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